1 Introduction

This article deals with the problem of representing the geometry of several (up to three) pinhole cameras. The idea that we put forward is that this can be done elegantly and conveniently using the formalism of the Grassmann-Cayley algebra. This formalism has already been presented to the Computer Vision community in a number of publications such as, for example, [Carlsson (1994), Faugeras and Mourrain (1995a)] but no effort has yet been made to systematically explore its use for representing the geometry of systems of cameras.

The thread that is followed here is to study the relations between the 3D world and its images obtained from one, two or three cameras as well as, when possible, the relations between those images with the idea of having an algebraic formalism that allows us to compute and estimate things while keeping the geometric intuition which, we think, is important. The Grassmann-Cayley or double algebra with its two operators join and meet that correspond to the geometric operations of summing and intersecting vector spaces or projective spaces was precisely invented to fill this need.

After a very brief introduction to the double algebra (more detailed contemporary discussions can be found for example in [Doubilet et al. (1974)] and [Barnabei et al. (1985)]), we apply the algebraic-geometric tools to the description of one pinhole camera in order to introduce such notions as the optical center, the projection planes and the projection rays which appear later. This introduction is particularly dense and only meant to make the paper more or less self-contained.

We then move on to the case of two cameras and give a simple account of the fundamental matrix [Longuet-Higgins (1981), Faugeras (1992), Carlsson (1994), Luong and Faugeras (1995)] which sheds some new light on its structure.

The next case we study is the case of three cameras. We present a new way of deriving the trifocal tensors which appear in several places in the literature. It has been shown originally by Shashua [Shashua (1994)] that the coordinates of three corresponding points in three views satisfy a set of algebraic relations of degree 3 called the trilinear relations. It was later on pointed out by Hartley [Hartley (1994)] that those trilinear relations were in fact arising from a tensor that governed the correspondences of lines between three views and which he called the trifocal tensor. Hartley also correctly pointed out that this tensor had been used, if not formally identified as such, by researchers working on the problem of the estimation of motion and structure from line correspondences [Spetsakis and Aloimonos (1990)]. Given three views, there exist three such tensors and we introduce them through the double algebra.

Each tensor seems to depend upon 26 parameters (27 up to scale), these 26 parameters are not independent since the number of degrees of freedom of three...
views has been shown to be equal to 18 in the projective framework (33 parameters for the 3 perspective projection matrices minus 15 for an unknown projective transformation) [Luong and Vieville (1994)]. Therefore the trifocal tensor can depend upon at most 18 independent parameters and its 27 components must satisfy a number of algebraic constraints, some of them have been elucidated [Shashua and Werman (1995), Avidan and Shashua (1996)]. We have given a slightly more complete account of those constraints in [Faugeras and Papadopoulos (1998)], used them to parameterize the tensors minimally (i.e. with 18 parameters) and to design an algorithm for their estimation given line correspondences. In this paper we explore those constraints in great detail and prove that two particular simple subsets are sufficient for a tensor to arise from three camera (theorems 4 and 5).

We denote vectors and matrices with bold letters, e.g. x and P. The determinant of a square matrix A is noted det(A). When the matrix is defined by a set of vectors, e.g. A = [a1, a2, a3] we use | a1 a2 a3 |. The canonical basis of R3 is noted e_i; i = 1, 2, 3. When dealing with projective spaces, such as P^2 and P^3, we occasionally make the distinction between a projective point, e.g. x and one of its coordinate vectors, x. The dual of P^n, the set of projective points is the set of projective lines (n = 2) or the set of projective planes (n = 3), it is denoted by P^*n. Let a, b, c, d be four vectors of R^3. We will use in section 5.5 Cramer’s relation (see [Faugeras and Mourrain (1995b)]):

\[ \begin{vmatrix} b & c & d & a \end{vmatrix} = a \begin{vmatrix} b & c & d \end{vmatrix} = b \begin{vmatrix} a & c & d \end{vmatrix} = c \begin{vmatrix} a & b & d \end{vmatrix} = d \begin{vmatrix} a & b & c \end{vmatrix} = 0 \]

2 Grassman-Cayley algebra

Let E be a vector space of dimension 4 on the field R. The corresponding three-dimensional projective space is noted P^3. We consider ordered sets of k, k ≤ 4 vectors of E. Such ordered sets are called k-sequences. We first define an equivalence relation over the set of k-sequences as follows. Given two k-sequences a_1, ..., a_k and b_1, ..., b_k, we say that they are equivalent when, for every choice of vectors x_{k+1}, ..., x_n we have:

\[ | a_1 \cdot \ldots \cdot a_k \cdot x_{k+1} \cdot \ldots \cdot x_n |= | b_1 \cdot \ldots \cdot b_k \cdot x_{k+1} \cdot \ldots \cdot x_n | \]

(1)

That this defines an equivalence relation is immediate. An equivalence class under this relation is called an extensor of step k and is written as

\[ a_1 \nabla a_2 \nabla \ldots \nabla a_k \]

(2)

The product operator \nabla is called the join for reasons related to its geometric interpretation. Let us denote by G_k(E), 1 ≤ k ≤ 4 the vector set generated by all extensors of step k, i.e. by all linear combinations of terms like (2). It is clear from the definition that G_1(E) = E.

To be complete one defines G_0(E) to be equal to the field R. The dimension of G_k(E) is \( \binom{n}{k} \). The join operator corresponds to the union of projective subspaces of P(E). The exterior algebra is the direct sum of the vector spaces G_k, k = 0, ..., 4 with the join operator. For example, a point of P^3 is represented by a vector of E, its coordinate vector, or equivalently by a point of G_1(E). The join M_1 \nabla M_2 of two distinct points M_1 and M_2 is the line (M_1, M_2). Similarly, the join M_1 \nabla M_2 \nabla M_3 of three distinct points M_1, M_2, M_3 is the plane (M_1, M_2, M_3). It is an extensor of step 3. The set of extensors of step 3 represents the sets of planes of P^3.

Let us study in more detail the case of the lines of P^3. Lines are extensors of step 2 and are represented by six-dimensional vectors of G_2(E) with coordinates \( L_{ij}, 1 \leq i < j \leq 4 \) which satisfy the well-known the Plücker relation:

\[ L_{12}L_{34} - L_{13}L_{24} + L_{14}L_{23} = 0 \]

(3)

This equation allows us to define an inner product between two elements L and L' of G_2(R^4):

\[ [L | L'] = L_{12}L_{34} + L_{13}L_{24} - L_{14}L_{23} = L_{12}L_{34} - L_{13}L_{24} + L_{14}L_{23} \]

(4)

We will use this inner product when we describe the imaging of 3D lines by a camera in section 4. Not all elements of G_2(E) are extensors of step 2 and it is known that:

**Proposition 1** An element L of G_2(E) represents a line if and only if [L | L] is equal to 0.

To continue our program to define algebraic operations which can be interpreted as geometric operations on the projective subspaces of P(E) we define a second operator, called the meet, and noted \( \triangle \), on the exterior algebra G(E). This operator corresponds to the geometric operation of intersection of projective subspaces. If A is an extensor of step k and B is an extensor of step h, k + h ≥ 4, the meet A \( \triangle \) B of A and B is an extensor of step k + h − 4. For example, if \( \Pi_1 \) and \( \Pi_2 \) are non proportional extensors of step 3, i.e. representing two plane, their meet \( \Pi_1 \triangle \Pi_2 \) is an extensor of step 2, representing the line of intersection of the two planes. Similarly, if \( \Pi \) is an extensor of step 3 representing a plane and L an extensor of step 2 representing a line, the meet \( \Pi \triangle L \) is either 0 if L is contained in \( \Pi \) or an extensor of step 1 representing the point of intersection of L and \( \Pi \). Finally, if \( \Pi \) is an extensor of step 3, a plane, and M an extensor of step 1, a point, the meet \( \Pi \triangle M \) is an extensor of step 0, a real number, which turns out to be equal to \( \Pi^T M \), the scalar product of the usual vector representation of the plane \( \Pi \) with a coordinate vector of the point M which we note \( (\Pi, M) \). The connection between a plane as a vector in G_2(E) and the usual vector representation \( \Pi \) is through the Hodge
operator and can be found for example in [Barnabei et al. (1985)].

We also define a special element of $G_4(E)$, called the integral. Let $\{a_1, \ldots, a_4\}$ be a basis of $E$ such that $|a_1 \cdot \ldots \cdot a_4| = 1$ (the $4 \times 4$ determinant), a unimodal basis. The extensor $I = a_1 \wedge \ldots \wedge a_4$ is called the integral. In section 4 we will need the following property of the integral:

**Proposition 2** Let $A$ and $B$ be two extensors such that $\text{step}(A) + \text{step}(B) = 4$. Then

$$A \wedge B = (A \wedge B) \wedge I = |A \cdot B| \cdot I$$

The inner product (4) has an interesting interpretation in terms of the join $L \vee L'$, an extensor of step 4:

**Proposition 3** We have the following relation:

$$L \vee L' = [L \mid L'] e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

(5)

where $e_1, \ldots, e_4$ is the canonical basis of $R^4$. In section 3, we will use the following results on lines:

**Proposition 4** Let $L$ and $L'$ be two lines. If the two lines are represented as the joins of two points $A$ and $B$ and $A'$ and $B'$, respectively, then:

$$[L \mid L'] = |A \cdot B \cdot A' \cdot B'|$$

If the two lines are represented as the meets of two planes $P$ and $Q$ and $P'$ and $Q'$, then:

$$[L \mid L'] = |P \cdot Q \cdot P' \cdot Q'|$$

(6)

If one line is represented as the meet of two planes $P$ and $Q$ and the other as the join of two points $A'$ and $B'$, then:

$$[L \mid L'] = \langle P|A'\rangle \langle Q|B'\rangle - \langle Q|A'\rangle \langle P|B'\rangle$$

(7)

We will also use in section 4 the following result:

**Proposition 5** Let $L$ and $L'$ be two lines. The inner product $[L \mid L']$ is equal to 0 if and only if the two lines are coplanar.

### 3 Geometry of one view

We consider that a camera can be modeled accurately as a pinhole and performs a perspective projection. If we consider two arbitrary systems of projective coordinates, for the image and the object space, the relationship between 2-D pixels and 3-D points can be represented as a linear projective operation which maps points of $P^3$ to points of $P^2$. This operation can be described by a $3 \times 4$ matrix $P$, called the *perspective projection* matrix of the camera:

$$m = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \overset{\mathcal{P}}{\rightarrow} \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{Z} \end{bmatrix} = \mathcal{P}m$$

(8)

This matrix is of rank 3. Its nullspace is therefore of dimension 1, corresponding to a unique point of $P^3$, the optical center $C$ of the camera.

We give a geometric interpretation of the rows of the projection matrix. We use the notation:

$$\mathcal{P}^T = \left[ \mathcal{I}^T \mathcal{A}^T \Theta^T \right]$$

(9)

where $\mathcal{I}$, $\mathcal{A}$, and $\Theta$ are the row vectors of $\mathcal{P}$. Each of these vectors represent a plane in 3D. These three planes are called the *projection planes* of the camera. The projection equation (8) can be rewritten as:

$$x : y : z = (\mathcal{I} \cdot \mathcal{M}) : (\mathcal{A} \cdot \mathcal{M}) : (\Theta \cdot \mathcal{M})$$

where, for example, $(\mathcal{I} \cdot \mathcal{M})$ is the dot product of the plane represented by $\mathcal{I}$ with the point represented by $\mathcal{M}$. This relation is equivalent to the three scalar equations, of which two are independent:

$$x(\mathcal{A} \cdot \mathcal{M}) - y(\mathcal{I} \cdot \mathcal{M}) = 0$$

$$y(\Theta \cdot \mathcal{M}) - z(\mathcal{A} \cdot \mathcal{M}) = 0$$

$$z(\mathcal{A} \cdot \mathcal{M}) - x(\Theta \cdot \mathcal{M}) = 0$$

(10)

The planes of equation $(\mathcal{I} \cdot \mathcal{M}) = 0$, $(\mathcal{A} \cdot \mathcal{M}) = 0$ and $(\Theta \cdot \mathcal{M}) = 0$ are mapped to the image lines of equations $x = 0$, $y = 0$, and $z = 0$, respectively. We have the proposition:

**Proposition 6** The three projection planes of a perspective camera intersect the retinal plane along the three lines going through the first three points of the standard projective basis.

The optical center is the unique point $C$ which satisfies $\mathcal{P}C = 0$. Therefore this point is the intersection of the three planes represented by $\mathcal{I}$, $\mathcal{A}$, $\Theta$. In the Grassmann-Cayley formalism, it is represented by the meet of those three planes $\mathcal{I} \wedge \mathcal{A} \wedge \Theta$. This is illustrated in Fig. 1. Because of the definition of the meet operator, the projective coordinates of $C$ are the four $3 \times 3$ minors of matrix $\mathcal{P}$:

**Proposition 7** The optical center $C$ of the camera is the meet $\mathcal{I} \wedge \mathcal{A} \wedge \Theta$ of the three projection planes.

The three projection planes intersect along the three lines $\mathcal{I} \wedge \mathcal{A}$, $\mathcal{A} \wedge \Theta$ and $\Theta \wedge \mathcal{I}$ called the *projection rays*. These three lines meet at the optical center $C$ and intersect the retinal plane at the first three points $e_1$, $e_2$ and $e_3$ of the standard projective basis. Given a pixel $m$, its optical ray $(C, m)$ can be expressed very simply as a linear combination of the three projection rays:

**Proposition 8** The optical ray $L_m$ of the pixel $m$ of projective coordinates $(x, y, z)$ is given by:

$$L_m = x\mathcal{A} \wedge \Theta + y\Theta \wedge \mathcal{I} + z\mathcal{I} \wedge \mathcal{A}$$

(11)
We, which other representation, we see this, let us take the dot product:

\[ \langle x\Lambda - y\Gamma, M \rangle = x\langle \Lambda, M \rangle - y\langle \Gamma, M \rangle \]

but since \( x : y = \langle \Gamma, M \rangle : \langle \Lambda, M \rangle \), this expression is equal to 0. Therefore, the plane \( x\Lambda - y\Gamma \) contains the optical center \( \Gamma : \Lambda \Delta \Theta \). Similarly, the planes \( z\Gamma - x\Theta \) and \( y\Theta - z\Lambda \) also contain the optical ray \((C, m)\), which can therefore be found as the intersection of any of these two planes. Taking for instance the first two planes that we considered, we obtain:

\[ (x\Lambda - y\Gamma) \Delta (z\Gamma - x\Theta) = -x(x\Lambda \Delta \Theta + y\Theta \Delta \Gamma + z\Gamma \Delta \Lambda) \]

The scale factor \( x \) is not significant, and if it is zero, another choice of two planes can be made for the calculation. We conclude that the optical ray \( l_{01} = (C, m) \) is represented by the line \( x\Lambda \Delta \Theta + y\Theta \Delta \Gamma + z\Gamma \Delta \Lambda \) (see proposition 11) for another interesting interpretation of this formula. \( \square \)

We also have an interesting interpretation of matrix \( P^T \), which we give in the following proposition:

**Proposition 9** The transpose \( P^T \) of the perspective projection matrix defines a mapping from the set of lines of the retinal plane to the set of lines going through the optical center. This mapping associates to the line \( l \) represented by the vector \( l = [x, y, z]^T \) the plane \( P^T l = x\Gamma + y\Lambda + z\Theta \).

**Proof:** The fact that \( P^T \) maps planar lines to planes is a consequence of duality. The plane \( x\Gamma + y\Lambda + z\Theta \) contains the optical center since each projection plane contains it. \( \square \)

Having discussed the imaging of points, let us tackle the imaging of lines which plays a central role in subsequent parts of this paper. Given two 3-D points \( M_1 \) and \( M_2 \), the line \( L = M_1 \vee M_2 \) is an element of \( G_3(\mathbb{R}^4) \) represented by its Plücker coordinates. The image \( I \) of that line through a camera defined by the perspective projection matrix \( P \) is represented by the \( 3 \times 1 \) vector:

\[
I = PM_1 \times PM_2 = \left[ \langle \Lambda, M_1 \rangle \langle \Theta, M_2 \rangle - \langle \Theta, M_2 \rangle \langle \Lambda, M_1 \rangle, \langle \Theta, M_1 \rangle \langle T, M_2 \rangle - \langle T, M_2 \rangle \langle \Theta, M_1 \rangle, \langle T, M_1 \rangle \langle \Lambda, M_2 \rangle - \langle \Lambda, M_2 \rangle \langle T, M_1 \rangle \right]^T
\]

Equation (7) of proposition 4 allows us to recognize the inner products of the projection rays of the camera with the line \( L \):

\[
I \simeq \left[ [\Lambda \Delta \Theta \mid L], [\Theta \Delta \Gamma \mid L], [\Gamma \Delta \Lambda \mid L] \right]^T \tag{12}
\]

We can rewrite this in matrix form:

\[
I \simeq \hat{P} L \tag{13}
\]

where \( \hat{P} \) is the following \( 3 \times 6 \) matrix:

\[
\hat{P} = \begin{bmatrix}
\Lambda \Delta \Theta \\
\Theta \Delta \Gamma \\
\Gamma \Delta \Lambda
\end{bmatrix}
\]

The matrix \( \hat{P} \) plays for 3-D lines the same role that the matrix \( P \) plays for 3-D points. Equation (13) is thus equivalent to

\[
l_1 : l_2 : l_3 = [\Lambda \Delta \Theta \mid L] : [\Theta \Delta \Gamma \mid L] : [\Gamma \Delta \Lambda \mid L]
\]

We have the following proposition:

**Proposition 10** The pinhole camera also defines a mapping from the set of lines of \( \mathbb{P}^3 \) to the set of lines of \( \mathbb{P}^2 \). This mapping is an application from the projective space \( P(G_3(\mathbb{R}^4)) \) (the set of 3D lines) to the projective space \( P(G_3(\mathbb{R}^3)) \) (the set of 2D lines). It is represented by a \( 6 \times 4 \) matrix, noted \( \hat{P} \) whose row vectors are the Plücker coordinates of the projection rays of the camera:

\[
\hat{P} = \begin{bmatrix}
\Lambda \Delta \Theta \\
\Theta \Delta \Gamma \\
\Gamma \Delta \Lambda
\end{bmatrix}
\]

The image \( I \) of a 3D line \( L \) is given by:

\[
l_1 : l_2 : l_3 = [\Lambda \Delta \Theta \mid L] : [\Theta \Delta \Gamma \mid L] : [\Gamma \Delta \Lambda \mid L]
\]

The null space of this mapping contains the set of lines going through the optical center of the camera.
Proof: We have already proved the first part. Regarding the nullspace, if \( L \) is a 3D line such that \( \mathcal{P}L = 0 \), then \( L \) intersect all three projection rays of the camera and hence goes through the optical center. □

The dual interpretation is also of interest:

**Proposition 11** The \( 3 \times 6 \) matrix \( \mathcal{P}^T \) represents a mapping from \( \mathbb{P}^2 \) to the set of 3D lines, subset of \( \mathcal{P}(G_2(\mathbb{R}^4)) \), which associates to each pixel \( m \) its optical ray \( L_m \).

**Proof:** Since \( \mathcal{P} \) represents a linear mapping from \( G_2(\mathbb{R}^4) \) to \( G_2(\mathbb{R}^3) \), \( \mathcal{P}^T \) represents a linear mapping from the dual \( G_2(\mathbb{R}^3)^1 \) of \( G_2(\mathbb{R}^3) \) which we can identify to \( G_1(\mathbb{R}^3) \), to the dual \( G_2(\mathbb{R}^4)^1 \) which we can identify to \( G_2(\mathbb{R}^4) \) (see for example [Barnabei et al. (1985)]) for the definition of the Hodge operator and duality). Hence it corresponds to a mapping from \( \mathbb{P}^2 \) to \( \mathcal{P}(G_2(\mathbb{R}^4)) \). If the pixel \( m \) has projective coordinates \( x, y \), and \( z \), we have:

\[
\mathcal{P}^T m \simeq x\Lambda \Delta \Theta + y\Theta \Delta \Gamma + z\Gamma \Delta \Lambda
\]

and we recognize the right hand side to be a representation of the optical ray \( L_m \). □

### 3.1 Affine digression

In the affine framework we can give an interesting interpretation of the third projection plane of the perspective projection matrix:

**Proposition 12** The third projection plane \( \Theta \) of the perspective projection matrix \( \mathcal{P} \) is the focal plane of the camera.

**Proof:** The points of the plane of equation \( \{ \Theta, M \} = 0 \) are mapped to the points in the retina plane such that \( z = 0 \). This is the equation of the line at infinity in the retina plane. The plane represented by \( \Theta \) is therefore the set of points in 3D space which do not project at finite distance in the retina plane. These points form the focal plane, which is the plane containing the optical center, and parallel to the retina plane. □

When the focal plane is the plane at infinity, i.e. \( \Theta \simeq e_4 \), the camera is called an affine camera and performs a parallel projection. Note that this class of cameras is important in applications, including the orthographic, weak perspective, and scaled orthographic projections.

### 4 Geometry of two views

In the case of two cameras, it is well-known that the geometry of correspondences between the two views can be described compactly by the fundamental matrix, noted \( F_{12} \) which associates to each pixel \( m_1 \) of the first view its epipolar line \( l_{m_1} \) in the second view:

\[
l_{m_1} \simeq F_{12} m_1
\]

similarly \( F_{21} = F_{12}^T \) associates to a pixel \( m_2 \) of the second view its epipolar line \( l_{m_2} \) in the first one.

The matrix \( F_{12} \) (resp. \( F_{21} \)) is of rank 2, the point in its null-space is the epipole \( e_{1,2} \) (resp. the epipole \( e_{2,1} \)):

\[
F_{12} e_{1,2} = F_{21} e_{2,1} = 0
\]

There is a very simple and natural way of deriving the fundamental matrix in the Grassman-Cayley formalism. We use the simple idea that two pixels \( m \) and \( m' \) are in correspondence if and only if their optical rays \( (C, m) \) and \( (C', m') \) intersect. We then write down this condition using propositions 5 and obtain the fundamental matrix using the properties of the double algebra.

We will denote the rows of \( \mathcal{P} \) by \( \Gamma, \Lambda, \Theta \), and the rows of \( \mathcal{P}' \) by \( \Gamma', \Lambda', \Theta' \). We have the following proposition:

**Proposition 13** The expression of the fundamental matrix \( F \) as a function of the row vectors of the matrices \( \mathcal{P} \) and \( \mathcal{P}' \) is:

\[
F = \begin{bmatrix}
\Lambda \Theta \Lambda' \Theta' \\
\Theta \Theta' \Gamma' \\
\Theta \Theta' \Gamma \\
\Lambda \Theta \Lambda' \Lambda' \\
\Theta \Gamma' \Lambda' \\
\Theta \Gamma' \Gamma
\end{bmatrix}
\]

(15)

**Proof:** Let \( m \) and \( m' \) be two pixels. They are in correspondence if and only if their optical rays \( (C, m) = L_m \) and \( (C', m') = L_{m'} \) intersect. According to proposition 5, this is equivalent to the fact that the inner product \( [L_m] [L_{m'}]^T \) of the two optical rays is equal to 0. Let us translate this algebraically. Let \( (x, y, z) \) (resp. \( (x', y', z') \)) be the coordinates of \( m \) (resp. \( m' \)). Using proposition 11, we write:

\[
L_m \simeq \mathcal{P}^T m \simeq x\Lambda \Delta \Theta + y\Theta \Delta \Gamma + z\Gamma \Delta \Lambda
\]

and:

\[
L_{m'} \simeq \mathcal{P}^T m' \simeq x'\Lambda' \Delta \Theta' + y'\Theta' \Delta \Gamma' + z'\Gamma' \Delta \Lambda'
\]

We now want to compute \( [L_m] [L_{m'}]^T \). In order to do this, we use proposition 3 and compute \( L_m \preceq L_{m'} \):

\[
L_m \preceq L_{m'} \simeq (x\Lambda \Delta \Theta + y\Theta \Delta \Gamma + z\Gamma \Delta \Lambda) \preceq (x'\Lambda' \Delta \Theta' + y'\Theta' \Delta \Gamma' + z'\Gamma' \Delta \Lambda')
\]

Using the linearity of the join operator, we obtain an expression which is bilinear in the coordinates of \( m \) and \( m' \) and contains terms such as:

\[
(x \Lambda \Delta \Theta) \preceq (x' \Lambda' \Delta \Theta')
\]

Since \( \Lambda \Delta \Theta \) and \( \Lambda' \Delta \Theta' \) are extensors of step 2, we can apply proposition 2 and write:

\[
(x \Lambda \Delta \Theta) \preceq (x' \Lambda' \Delta \Theta') = \Lambda \Theta \Lambda' \Theta' \preceq I
\]

where \( I \) is the integral defined in section 2. We have similar expressions for all terms in \( L_m \preceq L_{m'} \). We thus obtain:

\[
L_m \preceq L_{m'} = (m'^T F m) I
\]
where the $3 \times 3$ matrix $F$ is defined by equation (15). Since $L_m \mathcal{V} L_{m'}^T = [L_m \mid L_{m'}] I$, the conclusion follows. \(\square\)

Let us determine the epipoles in this formalism. We have the following simple proposition:

**Proposition 14** The expression of the epipoles $e$ and $e'$ as a function of the row vectors of the matrices $\mathcal{P}$ and $\mathcal{P}'$ is:

$$
e \approx \begin{bmatrix}
\Gamma' \Lambda' \Theta' & \mid \\
\Lambda' \Gamma' \Theta' & \mid \\
\Theta' \Gamma' \Lambda' & 
\end{bmatrix}
e'
\approx \begin{bmatrix}
\Gamma' \Lambda \Theta & \\
\Lambda' \Gamma \Theta & \\
\Theta' \Gamma \Lambda & 
\end{bmatrix}
$$

**Proof**: We have seen previously that $e$ (resp. $e'$) is the image of $C'$ (resp. $C$) by the first (resp. the second) camera. According to proposition 7, these optical centers are represented by the vectors of $G_1(R^4)$ $\Gamma \Delta \Lambda \Theta$ and $\Gamma' \Delta \Lambda' \Theta'$. Therefore we have, for example, that the first coordinate of $e$ is:

$$(\Gamma, \Gamma' \Delta \Lambda' \Theta')$$

which is equal to $$(\Gamma' \Delta \Lambda' \Theta') \Delta \Gamma = - | \Gamma \Gamma' \Lambda' \Theta'| \quad \text{\(\square\)}$$

4.1 Another affine digression

Let us now assume that the two cameras are affine cameras, i.e. $\Theta \simeq \Theta'$ $\simeq e_3$ (in fact it is sufficient that $\Theta \simeq \Theta'$). Because of the standard properties of determinants, it is clear from equation (15) that the fundamental matrix takes a special form:

**Proposition 15** The fundamental matrix of two affine cameras has the form:

$$
F = \begin{bmatrix}
0 & 0 & \Gamma \Delta \Lambda \Theta' \\
0 & 0 & \Gamma \Delta \Theta \Gamma' \\
\Lambda \Theta \Gamma' \Lambda' & \Theta \Gamma \Gamma' \Lambda' & \Gamma \Lambda \Gamma' \Lambda' & 
\end{bmatrix}
$$

(16)

5 Geometry of three views

5.1 Trifocal geometry from binocular geometry

When we add one more view, the geometry becomes more intricate, see figure 2. Note that we assume that the three optical centers $C_1$, $C_2$, $C_3$ are different, and call this condition the **general viewpoint assumption**. When they are not aligned they define a plane, called the **trifocal plane**, which intersects the three image planes along the **trifocal lines** $t_1$, $t_2$, $t_3$ which contain the epipoles $e_{i,j}$, $i \neq j; i = 1, \ldots, 3$, $j = 1, \ldots, 3$. The three fundamental matrices $F_{12}$, $F_{23}$ and $F_{31}$ are not independent since they must satisfy the three constraints:

$$e_{2,3}^T F_{12} e_{1,3} = e_{3,1}^T F_{23} e_{2,1} = e_{1,2}^T F_{31} e_{3,2} = 0 \quad (17)$$

which arise naturally from the trifocal plane: for example, the epipolar line in view 2 of the epipole $e_{1,3}$ is represented by $F_{12} e_{1,3}$ and is the image in view 2 of the optical ray $(C_1, e_{1,3})$ which is identical to the line $(C_1, C_3)$. This image is the trifocal line $t_2$ which goes through $e_{2,3}$, see figure 2, hence the first equation in (17).

This has an important impact on the way we have to estimate the fundamental matrices when three views are available: very efficient and robust algorithms are now available to estimate the fundamental matrix between two views from point correspondences [Zhang et al. (1995), Torr and Zisserman (1997), Hartley (1995)]. The constraints (17) mean that these algorithms cannot be used blindly to estimate the three fundamental matrices independently because the resulting matrices will not satisfy the constraints causing errors in further processes such as prediction.

Indeed, one of the important uses of the fundamental matrices in trifocal geometry is the fact that they in general allow to **predict** from two correspondences, say $(m_1, m_2)$ where the point $m_3$ should be in the third image: it is simply at the intersection of the two epipolar lines represented by $F_{13} m_1$ and $F_{23} m_2$, when this intersection is well-defined.

It is not well-defined in two cases:

1. In the general case where the three optical centers are not aligned, when the 3D points lie in the trifocal plane (the plane defined by the three optical centers), the prediction with the fundamental matrices fails because, in the previous example both epipolar lines are equal to the trifocal line $t_3$.

2. In the special case where the three centers are aligned, the prediction with the fundamental matrices fails always since, for example, $F_{13} m_1 = F_{23} m_2$, for all corresponding pixels $m_1$ and $m_2$ in views 1 and 2, i.e. such that $m_2^T F_{12} m_1 = 0$.

For these two reasons, as well as for the estimation
problem mentioned previously, it is interesting to characterize the geometry of three views by another entity, 
the trifocal tensor.

The trifocal tensor is really meant at describing line correspondences and, as such, has been well-
known under disguise in the part of the computer vision community dealing with the problem of structure 
from motion [Spetsakis and Aloimonos (1990b), Spetsakis and Aloimonos (1990a), Weng et al. (1992)] be-
fore it was formally identified by Hartley and Shashua [Hartley (1994), Shashua (1995)].

5.2 The trifocal tensors

Let us consider three views, with projection matrices \( \mathcal{P}_i, n = 1, 2, 3 \), a 3D line \( L \) with images \( l_i \). Given two images \( l_j \) and \( l_k \) of \( L \), \( L \) can be defined as the intersection (the meet) of the two planes \( \mathcal{P}^T_i l_j \) and \( \mathcal{P}^T_k l_k \):

\[
L \simeq \mathcal{P}^T_i l_j \cap \mathcal{P}^T_k l_k
\]

The vector \( L \) is the 6 \( \times \) 1 vector of Plücker coordinates of the line \( L \).

Let us write the right-hand side of this equation explicitly in terms of the row vectors of the matrices \( \mathcal{P}_j \) and \( \mathcal{P}_k \) and the coordinates of \( l_j \) and \( l_k \):

\[
L \simeq (l^T_j \Gamma_j + l^T_k \Lambda_j + l^T_k \Theta_j) \Delta (l^T_k \Gamma_k + l^T_k \Lambda_k + l^T_k \Theta_k)
\]

By expanding the meet operator in the previous equation, it can be rewritten in the following less compact form with the advantage of making the dependency on the projection planes of the matrices \( \mathcal{P}_j \) and \( \mathcal{P}_k \) explicit:

\[
L \simeq \left[ \begin{array}{cccc}
\Gamma_j & \Delta & \Gamma_k & \Delta & \Gamma_j & \Delta & \Theta_k & \Delta \\
\Lambda_j & \Delta & \Lambda_k & \Delta & \Lambda_j & \Delta & \Theta_k & \Delta \\
\Theta_j & \Delta & \Lambda_k & \Theta_j & \Delta & \Theta_k & \Delta & \Theta_k \\
\end{array} \right] l_k
\]

This equation should be interpreted as giving the Plücker coordinates of \( L \) as a linear combination of the lines defined by the meets of the projection planes of the perspective matrices \( \mathcal{P}_j \) and \( \mathcal{P}_k \), the coefficients being the products of the projective coordinates of the lines \( l_j \) and \( l_k \).

The image \( l_i \) of \( L \) is therefore obtained by applying the matrix \( \mathcal{P}_i \) (defined in section 3) to the Plücker coordinates of \( L \), hence the equation:

\[
l_i \simeq \mathcal{P}_i (\mathcal{P}^T_j l_j \cap \mathcal{P}^T_k l_k)
\]

which is valid for \( i \neq j \neq k \). Note that if we exchange view \( j \) and view \( k \), we just change the sign of \( \iota \), and therefore we do not change \( \iota \). A geometric interpretation of this is shown in figure 3. For convenience, we rewrite equation (19) in a more compact form:

\[
l_i \simeq T_i (l_j, l_k)
\]

This expression can be also put in a slightly less compact form with the advantage of making the dependency on the projection planes of the matrices \( \mathcal{P}_n \), \( n = 1, 2, 3 \) explicit:

\[
l_i \simeq \left[ l^T_j G^T_1 l_k \right] \left[ l^T_j G^T_2 l_k \right] \left[ l^T_j G^T_3 l_k \right]^T
\]

This is, in the projective framework, the exact analog of the equation used in the work of Spetsakis and Aloi-
monos [Spetsakis and Aloimonos (1990b)] to study the structure from motion problem from line correspond-
ences.

The three \( 3 \times 3 \) matrices \( G^T_n \), \( n = 1, 2, 3 \) are obtained from equations (18) and (19):

\[
G^T_i = \left[ \begin{array}{cccc}
| \Lambda_i & \Theta_i & \Gamma_i & \Theta_i \\
| \Lambda_i & \Theta_i & \Gamma_i & \Theta_i \\
| \Lambda_i & \Theta_i & \Gamma_i & \Theta_i \\
| \Lambda_i & \Theta_i & \Gamma_i & \Theta_i \\
\end{array} \right]
\]

Note that equation (19) allows us to predict the coordinates of a line \( l_i \) in image \( i \) given two images \( l_j \) and \( l_k \) of an unknown 3D line \( L \) in images \( j \) and \( k \), except in two cases where \( T_i (l_j, l_k) = 0 \):

1. When the two planes determined by \( l_j \) and \( l_k \) are identical i.e. when \( l_j \) and \( l_k \) are corresponding epipolar lines between views \( j \) and \( k \). This is equivalent to saying that the 3D line \( L \) is in an epipolar plane of the camera pair \( (j, k) \). The meet that appears in equation (19) is then 0 and the line \( l_i \) is undefined, see figure 4. If \( L \) is not in an epipolar plane of the camera pair \( (i, j) \) then we can use the equation:

\[
l_k \simeq \mathcal{P}_k (\mathcal{P}^T_j l_j \cap \mathcal{P}^T_j l_j)
\]

Figure 3: The line \( l_i \) is the image by camera \( i \) of the 3D line \( L \) intersection of the planes defined by the optical centers of the cameras \( j \) and \( k \) and the lines \( l_j \) and \( l_k \), respectively.
prediction is not possible by any of the formulas such as (19).

2. When $l_j$ and $l_k$ are epipolar lines between views $i$ and $j$ and $i$ and $k$, respectively. This is equivalent to saying that they are the images of the same optical ray in view $i$ and that $l_i$ is reduced to a point (see figure 5).

![Figure 4](image.png)

**Figure 4:** When $l_j$ and $l_k$ are corresponding epipolar lines, the two planes $P_j^i l_j$ and $P_k^i l_k$ are identical and therefore $T_i(l_j, l_k) = 0$.

![Figure 5](image.png)

**Figure 5:** When $l_j$ and $l_k$ are epipolar lines with respect to view $i$, the line $l_i$ is reduced to a point, hence $T_i(l_j, l_k) = 0$.

Except in those two cases, we have defined an application $T_i$ from $\mathbb{P}^2 \times \mathbb{P}^2$, the Cartesian product of two duals of the projective plane, into $\mathbb{P}^2$. This application is represented by an application $T_i$ from $\mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}^3$. This application is bilinear and antisymmetric and is represented by the three matrices $G_i^n$, $n = 1, 2, 3$. It is called the trifocal tensor for view $i$. The properties of this application can be summarized in the following theorem:

**Theorem 1** The application $T_i : \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is represented by the bilinear application $T_i$ such that $T_i(l_j, l_k) = P_i^{ij} l_j \Delta P_i^{ik} l_k$. $T_i$ has the following properties:

1. It is equal to 0 iff
   (a) $l_j$ and $l_k$ are epipolar lines with respect to the $i$th view, or
   (b) $l_j$ and $l_k$ are corresponding epipolar lines with respect to the pair $(j, k)$ of cameras.

2. Let $l_k$ be an epipolar line with respect to view $i$ and $l_i$ the corresponding epipolar line in view $i$, then for all lines $l_j$ in view $j$ which are not epipolar lines with respect to view $i$: $T_i(l_j, l_k) \neq 0$.

3. Similarly, let $l_j$ be an epipolar line with respect to view $i$ and $l_i$ the corresponding epipolar line in view $i$, then for all lines $l_k$ in view $k$ which are not epipolar lines with respect to view $i$: $T_i(l_j, l_k) \neq 0$.

4. If $l_j$ and $l_k$ are non corresponding epipolar lines with respect to the pair $(j, k)$ of views, then $T_i(l_j, l_k) = 0$, the trifocal line of the $i$th view, if the optical centers are not aligned 0 otherwise.

**Proof:** We have already proved point 1. In order to prove point 2, we notice that when $l_k$ is an epipolar line with respect to view $i$, the line $L$ is contained in an epipolar plane for the pair $(i, k)$ of cameras. Two cases can happen. If $L$ goes through $C_i$, i.e. if $l_j$ is an epipolar line with respect to view $i$, then $l_i$ is reduced to a point and this is point 1.a of the theorem. If $L$ does not go through $C_i$, $l_j$ is not an epipolar line with respect to view $i$ and the image of $L$ in view $i$ is independent of its position in the epipolar plane for the pair $(i, k)$, it is the epipolar line $l_i$ corresponding to $l_k$. The proof of point 3 is identical after exchanging the roles of cameras $k$ and $j$.

If $l_j$ and $l_k$ are non corresponding epipolar lines for the pair $(j, k)$ of views, the two planes $(C_j, l_j)$ and $(C_k, l_k)$ intersect along the line $(C_j, C_k)$. Thus, if $C_i$ is not on that line, its image $l_i$ in view $i$ is indeed the trifocal line $t_i$, see figure 2. □

A more pictorial view is shown in figure 6: the tensor is represented as a $3 \times 3$ cube, the three horizontal planes representing the matrices $G_i^n$, $n = 1, 2, 3$. It can be thought of as a black box which takes as its input two lines, $l_j$ and $l_k$, and outputs a third one, $l_i$. Hartley has shown [Hartley (1994), Hartley (1997)] that the trifocal tensors can be very simply parameterized by the perspective projection matrices $P_n$, $n = 1, 2, 3$ of the three cameras. This result is summarized in the following proposition:

**Proposition 16** (Hartley) Let $P_n$, $n = 1, 2, 3$ be the three perspective projection matrices of three cameras in general viewing position. After a change of coordinates, those matrices can be written, $P_1 = [I, 0]$, $P_2 = [X e_{2,1}]$ and $P_3 = [Y e_{3,1}]$ and the matrices
The corresponding lines in view $i$ are represented by $e_{n} \times e_{i,k}$ and can be obtained as $T_{i}(l_{j}^{n}, l_{k}^{i})$, $n = 1, 2, 3$ for any $l_{j}$ not equal to $l_{k}^{i}$ (see proposition 19).

**Proof:** The nullspace of $G_{i}^{n}$ is the set of lines $l_{k}^{i}$ such that $T_{i}(l_{j}^{n}, l_{k}^{i})$ has a zero in the $n$-th coordinate for all lines $l_{j}$. The corresponding lines $l_{j}$ such that $l_{j} = T_{i}(l_{j}^{n}, l_{k}^{i})$ all go through the point represented by $e_{n}, n = 1, 2, 3$ in the $i$-th retinal plane. This is true if and only if $l_{i,k}^{l}$ is the image in the $k$-th retinal plane of the projection ray $A_{i} \Delta \Theta_{i} (n = 1), \Theta_{i} \Delta \Gamma_{i} (n = 2)$ and $\Gamma_{i} \Delta \Lambda_{i} (n = 3)$: $l_{i,k}^{l}$ is an epipolar line with respect to view $i$ and theorem 1, point 2, shows that for each $n$ the corresponding line in view $i$ is independent of $l_{j}$. Moreover, it is represented by $e_{n} \times e_{i,k}$. □

A similar reasoning applies to the matrices $G_{i}^{nT}$:

**Proposition 19 (Hartley)** The nullspaces of the matrices $G_{i}^{nT}$ are the three epipolar lines, noted $l_{i}^{p}, n = 1, 2, 3$, in the $j$-th retinal plane of the three projection rays of camera $i$. These three lines intersect at the epipole $e_{i,j}$, see figure 7. The corresponding epipolar lines in view $i$ are represented by $e_{n} \times e_{i,j}$ and can be obtained as $T_{i}(l_{i}^{n}, l_{j})$, $n = 1, 2, 3$ for any $l_{k}$ not equal to $l_{k}^{i}$.

![Figure 6: A three-dimensional representation of the trifocal tensor.](image)

**Figure 6:** A three-dimensional representation of the trifocal tensor.

$G_{i}^{n}$ can be expressed as:

$$G_{i}^{n} = e_{i,j}^{T} Y^{(n)} - X^{(n)} e_{i,j}^{T}, n = 1, 2, 3 \quad (23)$$

where the vectors $X^{(n)}$ and $Y^{(n)}$ are the column vectors of the matrices $X$ and $Y$, respectively.

We use this proposition as a definition:

**Definition 1** Any tensor of the form $(23)$ is a trifocal tensor.

### 5.3 A third affine digression

If the three cameras are affine, i.e. if $\Theta_{i} \simeq \Theta_{j} \simeq \Theta_{k} \simeq e_{i}$, then we can read off equation $(22)$ the form of the matrices $G_{i}^{n}, n = 1, 2, 3$.

**Proposition 17** For affine cameras, the trifocal tensor takes the simple form:

$$G_{i}^{n} = \left[ \begin{array}{ccc} \Lambda_{i, \Theta, \Gamma, \Lambda_{i}} & 0 \\ \Lambda_{i, \Theta, \Lambda_{i}} & 0 \\ \theta_{i} & 0 \\ \theta_{i} & 0 \end{array} \right] \quad (24)$$

**5.4 Algebraic and geometric properties of the trifocal tensors**

The matrices $G_{i}^{n}, n = 1, 2, 3$ have interesting properties which are closely related to the epipolar geometry of the views $j$ and $k$. We start with the following proposition, which was proved for example in [Hartley (1997)]. The proof hopefully gives some more geometric insight of what is going on:

**Proposition 18 (Hartley)** The matrices $G_{i}^{n}$ are of rank 2 and their nullspaces are the three epipolar lines, noted $l_{k}^{i}$ in view $k$ of the three projection rays of camera $i$. These three lines intersect at the epipole $e_{k,i}$.

![Figure 7: The lines $l_{i}^{p}$ (resp. $l_{i}^{k}$), $n = 1, 2, 3$ in the nullspaces of the matrices $G_{i}^{nT}$ (resp. $G_{i}^{n}$) are the images of the three projection rays of camera $i$. Hence, they intersect at the epipole $e_{i,j}$, (resp. $e_{k,i}$). The corresponding epipolar lines in camera $i$ are obtained as $T_{i}(l_{i}^{n}, l_{j})$ (resp. $T_{j}(l_{j}^{n}, l_{k}^{i})$) for $l_{k} \neq l_{i}^{i}$ (resp. $l_{i} \neq l_{k}^{j}$).

This provides a geometric interpretation of the matrices $G_{i}^{n}$: they represent mappings from the set of lines in view $k$ to the set of points in view $j$ located on the epipolar line $l_{i}^{j}$ defined in proposition 19. This mapping is geometrically defined by taking the intersection of the plane defined by the optical center of the $k$th camera and any line of its retinal plane with the $n$th projection ray of the $i$th camera and forming the image of this point in the $j$th camera. This point does not exist when the plane contains the projection ray. The corresponding line in the $k$th retinal plane is the epipolar line $l_{i}^{k}$ defined in proposition 18. Moreover, the three columns of $G_{i}^{n}$ represent three points which all belong to the epipolar line $l_{i}^{n}$.

Similarly, the matrices $G_{i}^{nT}$ represent mappings from the set of lines in view $j$ to the set of points in
view $k$ located on the epipolar line $l^*_k$.

Remark 1 It is important to note that the rank of the matrices $G^n$ cannot be less than 2. Consider for example the case $n = 1$. We have seen in proposition 18 that the nullspace of $G^1$ is the image of the projection ray $A_i = \Theta_i$ in view $k$. Under our general viewpoint assumption, this projection ray and the optical center $C_k$ define a unique plane unless it goes through $C_k$, a situation that can be avoided by a change of coordinates in the retina plane of the $i$th camera. Therefore there is a unique line in the right nullspace of $G^1$ and its rank is equal to 2. Similar reasoning apply to $G^2$ and $G^3$.

A question that will turn out to be important later is that of knowing how many distinct lines $l^*_k$ (resp. $l^*_h$) can there be. This is described in the following proposition:

**Proposition 20** Under the general viewpoint assumption, the rank of the matrices $[I_1^2 I_2^2 I_3^2]$ and $[I_1^2 I_2^2 I_3^2]$ is 2.

**Proof:** We know from propositions 18 and 19 that the the ranks are less than or equal to 2 because each triplet of lines intersect at an epipole. In order for the ranks to be equal to 1, we would need to have only one line in either retina plane. But this would mean that the three planes defined by $C_k$ (resp. $C_j$) and the three projection rays of the $i$th camera are identical which is impossible since $C_i \neq C_k$ (resp. $C_j \neq C_k$) and the three projection rays of the $i$th camera are not coplanar.

Algebraically, this implies that the three determinants $det(G^n)$, $n = 1, 2, 3$ are equal to 0. Another constraint implied by proposition 20 is that the 3 x 3 determinants formed with the three vectors in the nullspaces of the $G^n$, $n = 1, 2, 3$ (resp. of the $G^n$, $n = 1, 2, 3$) are equal to 0. It turns out that the applications $T_i$, $i = 1, 2, 3$ satisfy other algebraic constraints which are also important in practice.

The question of characterizing exactly the constraints satisfied by the tensors is of great practical importance for the problem of estimating the tensors from triplets of line correspondences (see [Faugeras and Papadopoulos (1998)]). To be more specific, we know that the tensor is equivalent to the knowledge of the three perspective projection matrices and that they depend upon 18 parameters. On the other hand a trifocal tensor depends upon 27 parameters up to scale, i.e. 26 parameters. To be more precise, this means that the set of trifocal tensors is a manifold of dimension 18 in the projective space of dimension 26. There must therefore exist constraints between the coefficients that define the tensor. Our next task is to discover some of those constraints and find subsets of them which characterize the trifocal tensors, i.e. that guarantee that they have the form (23).

To simplify a bit the notations, we will assume in the sequel that $i = 1, j = 2, k = 3$ and will ignore the $i$th index everywhere, e.g. denote $T_1$ by $T$.

We have already seen several such constraints when we studied the matrices $G^n$. Let us summarize those constraints in the following proposition:

**Proposition 21** Under the general viewpoint assumption, the trifocal tensor $T$ satisfies the three constraints, called the rank constraints:

$$rank(G^n) = 2 \implies det(G^n) = 0 \hspace{1cm} n = 1, 2, 3$$

The trifocal tensor $T$ satisfies the two constraints, called the epipolar constraints:

$$rank([I_1^2 I_2^2 I_3^2]) = rank([I_1^3 I_2^3 I_3^3]) = 2 \implies \left| I_1^2 I_2^2 I_3^2 \right| = \left| I_1^3 I_2^3 I_3^3 \right| = 0$$

Those five constraints which are clearly algebraically independent since the rank constraints say nothing about the way the kernels are related constrain the form of the matrices $G^n$.

We now show that the coefficients of $T$ satisfy nine more algebraic constraints of degree 6 which are defined as follows. Let $e_i, n = 1, 2, 3$ be the canonical basis of $\mathbb{R}^3$ and let us consider the four lines $T(e_{k_2}, e_{k_3}), T(e_{k_2}, e_{k_3}), T(e_{k_2}, e_{k_1})$ and $T(e_{k_2}, e_{k_1})$ where the indexes $k_2$ and $k_3$ (resp. $k_2$ and $k_3$) are different. For example, if $k_2 = k_3 = 1$ and $k_2 - k_3 = 2$, the four lines are the images in camera 1 of the four 3D lines $\Gamma_2 \Delta \Gamma_2, \Lambda_2 \Delta \Lambda_2, \Gamma_2 \Delta \Lambda_2$ and $\Lambda_2 \Delta \Lambda_2$.

These four lines can be chosen in nine different ways satisfy an algebraic constraint which is detailed in the following theorem which is proved in [Faugeras and Mourrain (1995b)].

**Theorem 2** The trifocal tensor $T$ satisfies the 9 algebraic constraints of degree 6, called the vertical constraints:

$$\begin{align*}
| & T_{k_2 k_3} T_{k_2 k_3} T_{k_2 k_3} T_{k_2 k_3} | - \\
| & T_{k_2 k_3} T_{k_2 k_3} T_{k_2 k_3} T_{k_2 k_3} | = 0
\end{align*}$$

(24)

where $T_{i m}$ represents the vector $T(e_i, e_m)$.

The reader can convince himself that if he takes any general set of lines, then equation (24) is in general not satisfied. For instance, let $I_i = e_i, i = 1, 2, 3, 4$. It is readily verified that the left hand side of (24) is equal to -2.

Referring to figure 6, what theorem 2 says is that if we take four vertical columns of the trifocal cube (shown as dashed lines in the figure) arranged in such a way that they form a prism with a square basis, then the expression (24) is equal to 0. This is the reason why we call these constraints the vertical constraints in the sequel.
It turns out that the same kind of relations hold for the other two principal directions of the cube (shown as solid lines of different widths in the same figure):

**Theorem 3** The trifocal tensor $T$ satisfies also the nine algebraic constraints, called the row constraints:

$$
| T_{k_1 k_2} T_{k_1 k_3} T_{k_1 t_1 t_2} | = 0, \\
| T_{k_1 k_2} T_{k_1 k_3} T_{k_1 t_1 k_3} | = 0,
$$

(25)

and the nine algebraic constraints, called the column constraints:

$$
| T_{t_1 t_2} T_{t_1 t_2} T_{t_1 t_2} | = 0, \\
| T_{t_1 t_2} T_{t_1 t_2} T_{t_1 t_2} | = 0,
$$

(26)

**Proof**: We do the proof for the first set of constraints which concern the columns of the matrices $G^w$. The proof is analogous for the other set concerning the rows.

The three columns of $G^w$ represent three points $G_i^w, k = 1, 2, 3$ of the epipolar line $l_2$ (see the discussion after proposition 19). To be concrete, let us consider the first two columns of $G^1$ and $G^2$, the proof is similar for the other combinations. We consider the two sets of points defined by $a_2 G_1 + b_2 G_2$ and $a_2 G_1^2 + b_2 G_2^2$. These two sets are in projective correspondence, the collineation being the identity. It is known that the line joining two corresponding points envelops a conic. It is easily shown that the determinant of the matrix defining this conic is equal to:

$$
| T_{1,1} T_{1,2} T_{2,2} | = 0, \\
| T_{2,1} T_{2,2} T_{2,2} | = 0.
$$

In order to show that this expression is equal to 0, we show that the conic is degenerate, containing two points. This result is readily obtained from a geometric interpretation of what is going on.

The point $a_2 G_1 + b_2 G_2$ is the image by $G^1$ of the line $a_2 e_1 + b_2 e_2$, i.e. the first set of points is the image by $G^1$ of the pencil of lines going through the point $e_3$. Using again the geometric interpretation of $G^1$, we realize that those points are the images in the second image of the points of intersection of the first projection ray $\Lambda_1 \Delta \Theta^1$ of the first camera with the pencil of planes going through the third projection ray $\Gamma_3 \Delta \Lambda_3$ of the third camera. Similarly, the second set of points is the image of the points of intersection of the second projection ray $\Theta \Delta \Gamma^1$ of the first camera with the pencil of planes going through the third projection ray $\Gamma_3 \Delta \Lambda_3$ of the third camera.

The lines joining two corresponding points of those two sets are thus the images of the lines joining the two points of intersection of a plane containing the third projection ray $\Gamma_3 \Delta \Lambda_3$ of the third camera with the first and the second projection rays, $\Lambda_1 \Delta \Theta^1$ and $\Theta_1 \Delta \Gamma_1$, of the first camera. This line lies in the third projection plane $\Theta_1$ of the first camera and in the plane $\Pi$ of the pencil. Therefore it goes through the point of intersection of the third projection plane $\Theta_1$ of the first camera and the third projection ray $\Theta_2 \Delta \Gamma_3$ of the third camera, see figure 8. In image two, all the lines going through two corresponding points go through the image of that point. A special case occurs when the plane $\Pi$ goes through the first optical center, the two points are identical to the epipole $e_{2,1}$, and the line joining them is not defined. Therefore the conic is reduced to the two points $e_{2,1}$ and the point of intersection of the two lines $(G^1_1, G^2_1)$ and $(G^1_2, G^2_2)$. This point is the image in the second camera of the point of intersection $\Gamma_1 \Delta \Gamma_3 \Delta \Lambda_3$ of the first projection plane, $\Gamma_1$, of the first camera with the third projection ray, $\Gamma_3 \Delta \Lambda_3$, of the third camera. □

The theorem draws our attention to three sets of three points, i.e. three triangles, which have some very interesting properties. The triangle that came up in the proof is the one whose vertices are the images of the points $A_1 = \Gamma_1 \Delta \Lambda_3 \Delta \Theta_3, B_1 = \Gamma_1 \Delta \Theta^1 \Delta \Gamma_3$ and $D_1 = \Gamma_1 \Delta \Gamma_3 \Delta \Lambda_3$. The other two triangles are those whose vertices are the images of the points $A_2 = \Lambda_1 \Delta \Lambda_3 \Delta \Theta_3, B_2 = \Lambda_1 \Delta \Theta^1 \Delta \Gamma_3$, $D_2 = \Lambda_1 \Delta \Gamma_3 \Delta \Lambda_3$ on one hand, and $A_3 = \Theta^1 \Delta \Lambda_3 \Delta \Theta_3, B_3 = \Theta^1 \Delta \Theta^1 \Delta \Gamma_3, D_3 = \Theta^1 \Delta \Gamma_3 \Delta \Lambda_3$ on the other.

Note that the three sets of vertices $A_1, A_2, A_3, B_1, B_2, B_3$ and $D_1, D_2, D_3$, are aligned on the three projection rays of the third camera and therefore their images are also aligned, the three lines $l_{21} = (a_1, a_2, a_3), l_{22} = (b_1, b_2, b_3)$ and $l_{23} = (d_1, d_2, d_3)$ converging to the epipole $e_{2,3}$, see figure 9. The corresponding epipolar lines in image 3 are represented by $l_{3i} = e_{3,2} \times e_i, i = 1, 2, 3$, respectively. Note that all

![Figure 8: A plane of the pencil of axis $\Gamma_3 \Delta \Lambda_3$ intersects the plane $\Gamma_1$ along a line going through the point $\Gamma_1 \Delta \Gamma_3 \Delta \Lambda_3$. The points $a(a_2, b_2)$ and $b(v_2, b_2)$ are the images by $G_1$ of $a_2 G_1 + b_2 G_2$ and $a_2 G_1^2 + b_2 G_2^2$, respectively.](image-url)
points \(a_i, b_i, d_i, \ i = 1, 2, 3\) can be expressed as simple functions of the columns of the matrixes \(G^n\). For example:

\[
a_1 = (G^1_1 \times G^1_2) \times (G^1_2 \times G^2_2)
\]

Figure 9: The three triangles have corresponding vertices aligned on epipolar lines for the pair \((2, 3)\) of images.

The same is true of the constraints on the rows of the matrices \(G^n\). More specifically the constraints (26) introduce nine other points \((H_i, K_i, M_i), \ i = 1, 2, 3\) with \(H_1 = \Gamma_1 \Delta \Theta_2 \Delta \Theta_3, K_1 = \Gamma_1 \Delta \Theta_2 \Delta \Gamma_2, M_1 = \Gamma_1 \Delta \Gamma_2 \Delta \Theta_2, H_2 = \Lambda_1 \Delta \Lambda_2 \Delta \Theta_2, K_2 = \Lambda_1 \Delta \Theta_2 \Delta \Gamma_2, M_2 = \Lambda_1 \Delta \Gamma_2 \Delta \Lambda_2,\) and \(H_3 = \Theta_1 \Delta \Lambda_2 \Delta \Theta_2, K_3 = \Theta_1 \Delta \Theta_2 \Delta \Gamma_2, M_3 = \Theta_1 \Delta \Gamma_2 \Delta \Lambda_2\). The three sets of points \(H_1, H_2, H_3, K_1, K_2, K_3\) and \(M_1, M_2, M_3\) are aligned on the three projection rays of the second camera and therefore their images are also aligned, the three lines \(l_{21} = (h_1, h_2, h_3)\), \(l_{22} = (k_1, k_2, k_3)\) and \(l_{23} = (m_1, m_2, m_3)\) converging to the epipole \(e_{2,3}\).

The corresponding epipolar lines, \(l_{2i}, \ i = 1, 2, 3\) in image 3 are represented by \(e_{2,3} \times e_i, \ i = 1, 2, 3,\) respectively.

Note that this yields a way of recovering the fundamental matrix \(F_{23}\) since we obtain the two epipoles \(e_{2,3}\) and \(e_{3,2}\) and three pairs of corresponding epipolar lines, in fact six pairs. We will not address here the problem of recovering the epipolar geometry of the three views, let us simply mention the fact that the fundamental matrices which are recovered from the trifocal tensor are compatible in the sense that they satisfy the constraints (17).

There is a further set of constraints that are satisfied by any trifocal tensor and are also of interest. They are described in the next proposition:

**Proposition 22** The trifocal tensor \(T\) satisfies the ten algebraic constraints, called the extended rank constraints:

\[
\text{rank}(\sum_{n=1}^{3} \lambda_n G^n) \leq 2 \ \forall \lambda_n, n = 1, 2, 3
\]

**Proof**: The proof can be done either algebraically or geometrically. The algebraic proof simply uses the parameterization (23) and verifies that the constraints described in proposition 23 are satisfied. In the geometric proof one notices that for fixed values (not all zero) of the \(\lambda_n\)'s, and for a given line \(l_3\) in view 3, the point which is the image in view 2 of line \(l_3\) by \(\sum_{n=1}^{3} \lambda_n G^n\) is the image of the point defined by:

\[
\lambda_1 P_2^T l \Delta (\Lambda_1 \Delta \Theta_1) + \lambda_2 P_2^T l \Delta (\Theta_1 \Delta \Gamma_1) + \lambda_3 P_2^T l \Delta (\Gamma_1 \Delta \Lambda_1)
\]

This expression can be rewritten as:

\[
P_2^T l \Delta (\lambda_1 \Lambda_1 \Delta \Theta_1 + \lambda_2 \Theta_1 \Delta \Gamma_1 + \lambda_3 \Gamma_1 \Delta \Lambda_1)
\]

(27)

The line \(\lambda_1 \Lambda_1 \Delta \Theta_1 + \lambda_2 \Theta_1 \Delta \Gamma_1 + \lambda_3 \Gamma_1 \Delta \Lambda_1\) is an optical ray of the first camera (proposition 8), and when \(l\) varies in view 3, the point defined by (27) is well defined except when \(l\) is the image of that line in view 3. In that case the meet in (27) is zero and the image of that line is in the nullspace of \(\sum_{n=1}^{3} \lambda_n G^n\). □

Note that the proposition 22 is equivalent to the vanishing of the 10 coefficients of the homogeneous polynomial of degree 3 in the three variables \(\lambda_n, n = 1, 2, 3\) equal to \(\det(\sum_{n=1}^{3} \lambda_n G^n)\). The coefficients of the terms \(\lambda_n^3, n = 1, 2, 3\) are the determinants \(\det(G^3), n = 1, 2, 3\). Therefore the extended rank constraints contain the rank constraints.

To be complete, we give the expressions of the seven extended rank constraints which are different from the three rank constraints:

**Proposition 23** The seven extended rank constraints are given by:

\[
\lambda_1^2 \lambda_2 | G^1_1 G^2_1 G^3_1 | + | G^1_2 G^2_2 G^3_2 | + | G^1_3 G^2_3 G^3_3 | = 0
\]

(28)

\[
\lambda_1^2 \lambda_3 | G^1_1 G^2_1 G^3_1 | + | G^1_2 G^2_2 G^3_2 | + | G^1_3 G^2_3 G^3_3 | = 0
\]

(29)

\[
\lambda_2^2 \lambda_1 | G^1_1 G^2_1 G^3_1 | + | G^1_2 G^2_2 G^3_2 | + | G^1_3 G^2_3 G^3_3 | = 0
\]

(30)

\[
\lambda_2^2 \lambda_3 | G^1_1 G^2_1 G^3_1 | + | G^1_2 G^2_2 G^3_2 | + | G^1_3 G^2_3 G^3_3 | = 0
\]

(31)

\[
\lambda_3^2 \lambda_1 | G^1_1 G^2_1 G^3_1 | + | G^1_2 G^2_2 G^3_2 | + | G^1_3 G^2_3 G^3_3 | = 0
\]

(32)

\[
\lambda_3^2 \lambda_2 | G^1_1 G^2_1 G^3_1 | + | G^1_2 G^2_2 G^3_2 | + | G^1_3 G^2_3 G^3_3 | = 0
\]

(33)

\[
\lambda_1 \lambda_2 \lambda_3 | G^1_1 G^2_1 G^3_1 | + | G^1_2 G^2_2 G^3_2 | + | G^1_3 G^2_3 G^3_3 | = 0
\]

(34)
5.5 Constraints that characterize the tensor

We now show two results which are related to the question of finding subsets of constraints which are sufficient to characterize the trifocal tensors. These subsets are the implicit equations of the manifold of the trifocal tensors. The first result is given in the following theorem:

**Theorem 4** Let $T$ be a bilinear mapping from $\mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^2$ which satisfies the fourteen rank, epipolar and vertical constraints. Then this mapping is a trifocal tensor, i.e. it satisfies definition 1. Those fourteen algebraic equations are a set of implicit equations of the manifold of trifocal tensors.

The second result is that the ten extended constraints and the epipolar constraints characterize the trifocal tensors:

**Theorem 5** Let $T$ be a bilinear mapping from $\mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^2$ which satisfies the twelve extended rank and epipolar constraints. Then this mapping is a trifocal tensor, i.e. it satisfies definition 1. Those twelve algebraic equations are another set of implicit equations of the manifold of trifocal tensors.

The proof of those theorems will take us some time. We start with a proposition we will use to prove that the three rank constraints and the two epipolar constraints are not sufficient to characterize the set of trifocal tensors:

**Proposition 24** If a tensor $T$ satisfies the three rank constraints and the two epipolar constraints, then its matrices $G^n$, $n = 1, 2, 3$ can be written:

$$G^n = a_{n}X^{(n)}Y^{(n)T} + X^{(n)}_{2}Y^{(n)T}_{1} + e_{21}Y^{(n)T}_{1}, \quad (35)$$

where $e_{21}$ (resp. $e_{31}$) is a fixed point of image 2 (resp. of image 3), the three vectors $X^{(n)}$ represent three points of image 2, and the three vectors $Y^{(n)}$ represent three points of image 3.

**Proof:** The rank constraints allow us to write:

$$G^n = X^{(n)}_1Y^{(n)T}_1 + X^{(n)}_2Y^{(n)T}_2 \quad (36)$$

where the six vectors $X^{(n)}_1$, $X^{(n)}_2$ $n = 1, 2, 3$ represent six points of the second image and the six vectors $Y^{(n)}_1$, $Y^{(n)}_2$ $n = 1, 2, 3$ represent six points of the third image.

The right nullspace of $G^n$ is simply the cross-product $X^{(n)}_1 \times X^{(n)}_2$, the left nullspace being $Y^{(n)}_1 \times Y^{(n)}_2$. Those two sets of three nullspaces are of rank 2 (proposition 21). Let us consider the first set. We can write the corresponding matrix as:

$$[X^{(1)}_1 \times X^{(2)}_1 \times X^{(3)}_1 \times X^{(3)}_2] = Z_1 T_1^T + Z_2 T_2^T$$

With obvious notations, we have in particular:

$$X^{(1)}_1 \times X^{(1)}_2 = T_{11} Z_1 + T_{21} Z_2$$

Let us now interpret this equation geometrically: the line represented by the vector $X^{(1)}_1 \times X^{(1)}_2$, i.e. the line going through the points $X^{(1)}_1$ and $X^{(1)}_2$ belongs to the pencil of lines defined by the two lines represented by the vectors $Z_1$ and $Z_2$. Therefore it goes through their point of intersection represented by the cross-product $Z_1 \times Z_2$ and we write $X^{(1)}_1$ as a linear combination of $X^{(1)}_1$ and $Z_1 \times Z_2$:

$$X^{(1)}_1 = \alpha_1 X^{(1)}_1 + \beta_1 Z_1 \times Z_2$$

We write $e_{21}$ for $Z_1 \times Z_2$ and note that our reasoning is valid for $X^{(1)}_1$ and $X^{(2)}_n$:

$$X^{(2)}_n = \alpha_n X^{(n)}_1 + \beta_n e_{21}, \quad n = 1, 2, 3$$

The same exact reasoning can be applied to the pairs $Y^{(1)}_1$, $Y^{(2)}_1$, $n = 1, 2, 3$ yielding the expression:

$$Y^{(1)}_1 = \gamma_n Y^{(n)}_1 + \delta_n e_{31}$$

We have exchanged the roles of $Y^{(1)}_1$ and $Y^{(2)}_1$ for reasons of symmetry in the final expression of $G_n$. Replacing $X^{(2)}_n$ and $Y^{(1)}_1$ by their values in the definition (36) of the matrix $G_n$, we obtain:

$$G^n = (\alpha_n + \gamma_n)X^{(n)}_1Y^{(n)T}_1 + \beta_n X^{(n)}_1 e_{31} + \beta_n e_{21} Y^{(n)T}_1$$

We can absorb the coefficients $\delta_n$ in $X^{(n)}$, the coefficients $\beta_n$ in $Y^{(n)}$ and we obtain the announced relation. □

The next proposition is a proof of theorem 4 that the fourteen rank and epipolar constraints characterize the set of trifocal tensors:

**Proposition 25** Let $T$ be a bilinear mapping from $\mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^2$ which satisfies the fourteen rank, epipolar and vertical constraints. Then its matrices $G^n$ take the form:

$$G^n = e_{21} Y^{(n)T} + X^{(n)T} e_{31} \quad (37)$$

**Proof:** In order to show this, we note that the nine vertical constraints imply that $T(l_{11}, l_{12}) = 0$ for all pair of epipolar lines $(l_{12}, l_{13})$, i.e. for all pairs of lines such that $l_{12}$ contains the point $e_{21}$ and $l_{13}$ contains the point $e_{31}$ defined in (24). Indeed, this implies that $a_n(l_{12}^T X^{(n)}) \cdot (Y^{(n)T} l_{13}) = 0$ for all pairs of epipolar lines $(l_{12}, l_{13})$ which implies $a_n = 0$ unless either $X^{(n)}$ is identical to $e_{21}$ or $Y^{(n)}$ is identical to $e_{31}$ which contradicts the hypothesis that the rank of $G^n$ is two.

In order to show this it is sufficient to show that each of the nine constraints implies that $T(l_{11}, l_{13_i}) = 0$, $i, j = 1, 2, 3$ where $l_{12}$, (resp. $l_{13}$) is an epipolar line for the pair $(1, 2)$ (resp. the pair $(1, 3)$) of cameras,
going the ith (resp. the jth) point of the canonical basis. This is sufficient because we can assume that, for example, $e_{2,1}$ does not belong to the line represented by $e_3$. In that case, any epipolar line $l_{21}$ can be represented as a linear combination of $l_{311}$ and $l_{312}$:

$$l_{21} = \alpha_3 l_{311} + \beta_3 l_{312}$$

Similarly, any epipolar line $l_{31}$ can be represented as a linear combination of $l_{311}$ and $l_{312}$, given that $e_{3,1}$ does not belong to the line represented by $e_3$:

$$l_{31} = \alpha_3 l_{311} + \beta_3 l_{312}$$

The bilinearity of $T$ allows us to conclude that $T(l_{21}, l_{31}) = 0$.

To simplify the notations we define:

$$\lambda_1 = T(e_{b_2}, e_{b_3}) \quad \lambda_2 = T(e_{e_2}, e_{e_3}) \quad \lambda_3 = T(e_{b_2}, e_{e_3}) \quad \lambda_4 = T(e_{e_2}, e_{e_3})$$

To help the reader follow the proof, we encourage him to take the example $b_2 = b_3 = 1$ and $l_2 = l_3 = 2$. If the tensor $T$ were a trifocal tensor, the four lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ would be the images of the 3D lines $\Gamma_2 \Delta \Gamma_3, \Lambda_2 \Delta \Gamma_3, \Gamma_2 \Delta \Lambda_3, \Lambda_2 \Delta \Lambda_3$, respectively.

We now consider the two lines $d_1$ and $d_2$ in image 1 which are defined as follows. $d_1$ goes through the point of intersection of $\lambda_1$ and $\Lambda_2$ (the image of the point $\Gamma_2 \Delta \Gamma_3$ and $\Lambda_2$) and the point of intersection of the lines $\lambda_3$ and $\Lambda_4$ (the image of the point $\Gamma_2 \Delta \Lambda_3$ and $\Lambda_4$). In our example, $d_1$ is the image of the projection ray $\Gamma_2 \Delta \Lambda_2$. $d_2$, on the other hand, goes through the point of intersection of $\lambda_1$ and $\Lambda_3$ (the image of the point $\Gamma_2 \Delta \Gamma_3$ and $\Lambda_3$) and the point of intersection of $\lambda_4$ and $\Lambda_3$ (the image of the point $\Gamma_2 \Delta \Lambda_3$ and $\Lambda_3$). In our example, $d_2$ is the image of the projection ray $\Gamma_2 \Delta \Lambda_3$. Using elementary geometry, it is easy to find:

$$d_1 = | \lambda_2 \lambda_3 \lambda_4 | \lambda_1 - | \lambda_1 \lambda_3 \lambda_4 | \lambda_2$$

$$d_2 = | \lambda_1 \lambda_3 \lambda_4 | \lambda_2 + | \lambda_1 \lambda_2 \lambda_3 | \lambda_4$$

According to the definition of the lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, $d_1$ is the image by $T$ of the two lines $(\lambda_1 \lambda_3 \lambda_4 | e_{b_2} - | \lambda_1 \lambda_3 \lambda_4 | e_{e_2}, e_{e_3})$ and $d_2$ the image by $T$ of the two lines $(\lambda_1 \lambda_3 \lambda_4 | e_{b_2} - | \lambda_1 \lambda_3 \lambda_4 | e_{e_2}, e_{e_3})$.

We now proceed to show that $T((\lambda_1 \lambda_3 \lambda_4 | e_{b_2} - | \lambda_1 \lambda_3 \lambda_4 | e_{e_2}, e_{e_3}) = 0$. Using the bilinearity of $T$, we have:

$$T((| \lambda_2 \lambda_3 \lambda_4 | e_{b_2} - | \lambda_1 \lambda_3 \lambda_4 | e_{e_2}, \lambda_1 \lambda_3 \lambda_4 | e_{b_2} - | \lambda_1 \lambda_3 \lambda_4 | e_{e_2}, e_{e_3}) = 0.$$
Proposition 26 Any bilinear mapping $T$ which satisfies the 14 rank, epipolar and vertical constraints also satisfies the 18 row and columns constraints.

Proof: The proof consists in noticing that if $T$ satisfies the rank, epipolar and vertical constraints, according to proposition 25, it satisfies definition 1 and therefore, according to theorem 3, it satisfies the row and column constraints.

The reader may wonder about the ten extended rank constraints: Are they sufficient to characterize the trilinear tensor? the following proposition answers this question negatively.

Proposition 27 The ten extended rank constraints do not imply the epipolar constraints.

Proof: The proof consists in exhibiting a counterexample. The reader can verify that the tensor $T$ defined by:

$$G^1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad G^2 = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

satisfies the ten extended rank constraints and that the corresponding three left nullspaces are the canonic lines represented by $e_n, n = 1, 2, 3$ which do not satisfy one of the epipolar constraints.

Before we prove theorem 5 we prove the following proposition:

Proposition 28 The three rank constraints and the two epipolar constraints do not characterize the set of trifocal tensors.

Proof: Indeed, proposition 24 gives us a parametrization of the matrices $G^n$ in that case. It can be verified that for such a parametrization, the vertical constraints are not satisfied. Assume now that the rank and epipolar constraints imply that the tensor is a trifocal tensor, then, according to proposition 2, it satisfies the vertical constraints, a contradiction.

We are now ready to prove theorem 5:

Proof: The proof consists in showing that any bilinear application $T$ that satisfies the five rank and epipolar constraints, i.e. whose matrices $G^n$ can be written as in (35) and the remaining seven extended rank constraints (28-34) can be written as in (37), i.e. is such that $a_n = 0, n = 1, 2, 3$.

If we use the parametrization (35) and evaluate the constraints (28-33), we find:

$$-a_2 \begin{bmatrix} e_{2,1} \ X^{(1)} \ X^{(2)} \ | \ e_{3,1} \ Y^{(1)} \ Y^{(2)} \end{bmatrix} \quad (38)$$

$$-a_3 \begin{bmatrix} e_{2,1} \ X^{(1)} \ X^{(3)} \ | \ e_{3,1} \ Y^{(1)} \ Y^{(3)} \end{bmatrix} \quad (39)$$

$$-a_1 \begin{bmatrix} e_{2,1} \ X^{(2)} \ X^{(3)} \ | \ e_{3,1} \ Y^{(1)} \ Y^{(2)} \end{bmatrix} \quad (40)$$

$$-a_3 \begin{bmatrix} e_{2,1} \ X^{(2)} \ X^{(3)} \ | \ e_{3,1} \ Y^{(2)} \ Y^{(3)} \end{bmatrix} \quad (41)$$

$$-a_1 \begin{bmatrix} e_{2,1} \ X^{(1)} \ X^{(3)} \ | \ e_{3,1} \ Y^{(1)} \ Y^{(3)} \end{bmatrix} \quad (42)$$

$$-a_2 \begin{bmatrix} e_{2,1} \ X^{(2)} \ X^{(3)} \ | \ e_{3,1} \ Y^{(2)} \ Y^{(3)} \end{bmatrix} \quad (43)$$

In those formulas, our attention is drawn to determinants of the form $| e_{2,1} \ X^{(i)} \ X^{(j)} |, i \neq j$ (type 2) and $| e_{3,1} \ Y^{(i)} \ Y^{(j)} |, i \neq j$ (type 3). The nullity of a determinant of the first type implies that the epipole $e_{2,1}$ (resp. $e_{3,1}$) is on the line defined by the two points $X^{(i)}, X^{(j)}$ (resp. $Y^{(i)}, Y^{(j)}$), if the corresponding points are distinct.

If all determinants are non zero, the constraints (38-43) imply that all $a_n$'s are zero. Things are slightly more complicated if some of the determinants are equal to 0.

We prove that if the matrices $G^n$ are of rank 2, no more than one of the three determinants of each of the two types can equal 0. We consider several cases.

The first case is when all points of one type are different. Suppose first that the three points represented by the three vectors $X^{(n)}$ are not aligned. Then, having two of the determinants of type 2 equal to 0 implies that the point $e_{2,1}$ is identical to one of the points $X^{(n)}$ since it is at the intersection of two of the lines they define. But, according to equation (35), this implies that the corresponding matrix $G^n$ is of rank 1, contradicting the hypothesis that this rank is 2. Similarly, if the three points $X^{(n)}$ are aligned, if one determinant is equal to 0, the epipole $e_{2,1}$ belongs to the line $(X^{(1)}, X^{(2)}, X^{(3)})$ which means that the three epipolar lines $l_{1}^{(1)}, l_{2}^{(2)}, l_{3}^{(3)}$ are identical contradicting the hypothesis that they form a matrix of rank 2. Therefore, in this case, all three determinants are non null.

The second case is when two of the points are equal, e.g. $X^{(1)} \simeq X^{(2)}$. The third point must then be different, otherwise we would only have one epipolar line contradicting the rank 2 assumption on those epipolar lines, and, if it is different, the epipole $e_{2,1}$ must not be on the line defined by the two points for the same reason. Therefore in this case also at most one of the determinants is equal to 0.

Having at most one determinant of type 2 and one of type 3 equal to 0 implies that at least two of the $a_n$ are 0. This is seen by inspecting the constraints (38-43).
If we now express the seventh constraint:

\[
\begin{align*}
\alpha_1 \alpha_2 \alpha_3 & \ | \ Y_1(1) Y_2(2) Y_3(3) \parallel X_1(1) X_2(2) X_3(3) | - \\
& \left( | e_{2,1} X_1(1) X_2(2) | || e_{3,1} Y_1(1) Y_2(2) | + \\
& | e_{3,1} Y_1(1) Y_2(2) || e_{2,1} X_1(1) X_2(2) | \alpha_1 + \\
& | e_{3,1} Y_1(1) Y_2(2) || e_{2,1} X_1(1) X_2(2) | \alpha_2 - \\
& | e_{2,1} X_1(1) X_2(2) | \alpha_3 + \\
& | e_{3,1} Y_1(1) Y_2(2) || e_{2,1} X_1(1) X_2(2) | \alpha_3 + \\
& | e_{2,1} X_1(1) X_2(2) | || e_{3,1} Y_1(1) Y_2(2) | \alpha_1 \alpha_3, \\
\end{align*}
\]

we find that it is equal to the third \(\alpha_n\) multiplied by two of the nonzero determinants, implying that the third \(\alpha_n\) is null and completing the proof.

Let us give a few examples of the various cases. Let us assume first that \(| e_{2,1} X_1(1) X_2(2) | = | e_{2,1} Y_1(1) Y_2(2) | = 0\). We find that the constraints (42), (43) and (41) imply \(\alpha_1 = \alpha_2 = \alpha_3 = 0\). The second situation occurs if we assume for example \(| e_{2,1} X_1(1) X_2(2) | = | e_{3,1} Y_1(1) Y_2(2) | = 0\). We find that the constraints (43) and (41) imply \(\alpha_2 = \alpha_3 = 0\). The constraint (34) takes then the form:

\[
- | e_{2,1} X_1(1) X_2(2) | || e_{3,1} Y_1(1) Y_2(2) | \alpha_1,
\]

and implies \(\alpha_1 = 0\).

Note that from a practical standpoint, theorem 5 provides a simple set of sufficient constraints than theorem 4: The ten extended constraints are of degree 3 in the elements of \(T\) whereas the nine vertical constraints are of degree 6 as are the two epipolar constraints.

This situation is more or less similar to the one with the \(E\)-matrix [Longuet-Higgins (1981)]. It has been shown in several places, for example in [Faugeras (1993)] proposition 7.2 and proposition 7.3, that the set of real \(E\)-matrices is characterized either by the two equations:

\[
\det(E) = 0 \quad \frac{1}{2} \text{Tr}^2(E E^T) - \text{Tr}((E E^T)^2) = 0,
\]

or by the nine equations:

\[
\frac{1}{2} \text{Tr}(EE^T)E - EE^TE = 0.
\]

In a somewhat analogous way, the set of trifocal tensors is characterized either by the fourteen, rank, epipolar and vertical constraints (theorem 4) or by the twelve extended rank and epipolar constraints (theorem 5).

6 Conclusion

We have shown a variety of applications of the Grassmann-Cayley or double algebra to the problem of modeling systems of up to three pinhole cameras. We have analyzed in detail the algebraic constraints satisfied by the trilinear tensors which characterize the geometry of three views. In particular, we have isolated two subsets of those constraints that are sufficiently to guarantee that a tensor that satisfies them arises from the geometry of three cameras. Each of those subsets is a set of implicit equations for the manifold of trifocal tensors. We have shown elsewhere [Faugeras and Papadopoulo (1998)] how to use some of those equations to parameterize the tensors and estimate them from line correspondences in three views.

References


[Shashua, 1994] Amnon Shashua. Trilinearity in visual recognition by alignment. In Eklundh [1994], pages 479–484.


