Abstract — This paper deals with the problem of expliciting the conditions under which a given state-dependent switching law stabilizes a given switched linear system. The given conditions rely on geometric interpretations and a simple Lyapunov function is found. The results are, therefore, extended to switched affine systems, where the concept of stability must be understood in a broader sense. The law is simulated in two-dimensional bimodal systems and it stabilizes a system that meets the explicit conditions, but without a strict Hurwitz combination. As a switched affine system example, a buck converter is simulated, showing an extreme case of overshoot suppression.

Keywords — Hybrid systems, Switched linear systems, switched affine systems, strict Hurwitz combination, stabilizing state-dependent switching law.

Resumo — Este artigo lida com o problema de explicitar as condições sob as quais uma simples lei de chaveamento dependente de estados estabiliza um dado sistema linear chaveado. As condições dadas contam com interpretações geométricas e uma simples função de Lyapunov é encontrada. Os resultados são, desta forma, estendidos a sistemas afins chaveados, onde o conceito de estabilidade deve ser compreendido de um sentido mais amplo. A lei é simulada em sistemas bidimensionais bimodais e estabiliza um sistema que atende às condições explicitadas, mas sem combinação Hurwitz estrita. Como um exemplo de sistema chaveado afim, um conversor buck é simulado, mostrando um caso extremo de supressão de sobre-sinal.

Palavras-chave — Sistemas híbridos, Sistemas lineares chaveados, sistemas afins chaveados, combinação Hurwitz estrita, lei de chaveamento estabilizante dependente de estados.

1 Introduction

Switched linear and switched affine systems are important subclasses of hybrid systems. In the last twenty years, a great advance was made in this field, as its applications are extensive, such as: switched mode power supplies (SMPS's) (Deaecto et al., 2010) and (Cardim et al., 2009), adaptive control (Narendra and Balakrishnan, 1997) and sliding modes, which may occur, in books like Khalil (1996) and Slotine and Li (1991). One last important comment is that the systems herein are essentially nonlinear. This way, it is important to review some facts on stability and sliding modes, which may occur, in books like Khalil (1996) and Slotine and Li (1991).
for bimodal two-dimensional linear systems without strict Hurwitz combinations. Section 3 studies the equilibrium point problem in switched affine systems and adapts the results from the previous section for these systems. Simulations are made for a buck converter. In section 4, the results are commented and conclusions are pointed out.

2 Switched Systems and Results for Switched Linear Systems

2.1 Characterization of the Switched Linear and Affine Systems

A $n$-dimensional $N$-modal switched affine system is described by (1).

\[
\dot{x} = A_{\sigma(x,t)}x + b_{\sigma(x,t)}
\]  

Where $x \in \mathbb{R}^n$ is the state, $\dot{x}$ its time derivative, and $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are the system matrices. Moreover, $\sigma(\cdot, \cdot)$ is the switching law which, in general form, can be written as $\sigma : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{N}_N$, and the codomain of this function is the set of the first $N$ strict natural numbers, that is, $\{1, 2, \cdots, N\}$.

If the switching law does not depend on time, that is, if $\sigma(x,0) = \sigma(x,t)$ for all $t \geq 0$, then the signal is said to be state-dependent. This is the class of switching signals that will be discussed in this paper. Other kinds of switching signals are extensively discussed in the literature, as in Liberzon (2003) and Sun and Ge (2005).

A switched linear system has the same structure as in (1), with $b = 0$. This will be the first kind of system to be addressed.

2.2 Results for Switched Linear Systems

All results in this paper are based on Lemma 1 which will prove itself useful.

\textbf{Lemma 1} Let (1) be a $n$-dimensional $N$-modal switched linear system (i.e., $b = 0$). The following statements are equivalent$^\dagger$:

(i) For each $x \in \mathbb{R}^n$, $\exists j \in \mathbb{N}_N$ such that $x' \cdot \dot{x} < 0$;

(ii) Given a ball$^\ddagger$ of radius $\rho$ around the origin, $B_\rho$, for each$^\S$ $x_\rho \in \partial B_\rho$, $\exists j \in \mathbb{N}_N$, such that $x_\rho' \cdot \dot{x}_\rho < 0$.

\textbf{Proof:} It is clear that (i) implies (ii). Therefore, it is only necessary to prove the converse.

Indeed, observe that any $x \in \mathbb{R}^n$ can be written as $x = \alpha x_\rho$, $\alpha \in \mathbb{R}_+^n$, and $x_\rho \in \partial B_\rho$. Then:

\[
\begin{align*}
\dot{x}' \cdot \dot{x} &= \alpha x_\rho' \cdot \frac{d\alpha x_\rho}{dt} = \alpha^2 x_\rho' \cdot \dot{x}_\rho < 0
\end{align*}
\]

The geometrical interpretation of Lemma 1 is quite simple and interesting: to verify that, for the entire state space, the time derivative of the state points (at least partially) towards the center, it is necessary and sufficient to verify that property on any chosen ball hull centered in the origin. Figure 1 illustrates this fact in $\mathbb{R}^2$. Now the Lemma is used to prove the validity of the proposed switching law, contained in Theorem 2.

\begin{figure}
\begin{center}
\includegraphics{fig1.png}
\caption{Figure 1: Geometric interpretation of Lemma 1.}
\end{center}
\end{figure}

\textbf{Theorem 2} Let (1) be a $n$-dimensional $N$-modal switched linear system and $B_\rho$ a ball of radius $\rho$ centered in the origin where, for each given $x \in \partial B_\rho$, $\exists j \in \mathbb{N}_N$, such that $x' \cdot \dot{x} < 0$. Then, the state-dependent switching law (2) makes the system globally asymptotically stable.

\[
\sigma(x) = \arg \min_{i \in \mathbb{N}_N} x' A_i x
\]  

\textbf{Proof:} Indeed, the function $V(x) = ||x||^2$ is a global Lyapunov function.

i) $V(x) \geq 0$ and $V(x) = 0$ if and only if $x = 0$, as it is an euclidean norm;

ii) $V(x) \to \infty$ as $||x|| \to \infty$;

iii) Let $k(x) = \arg \min_{i \in \mathbb{N}_N} x' A_i x$. Then, for any given $x \in \mathbb{R}^n$:

\[
\dot{V}(x) = \nabla V' \cdot \dot{x} = 2 x' A_{\sigma(x)} x = 2 \min_{i \in \mathbb{N}_N} x' A_i x \\
\leq 2 x' A_{k(x)} x < 0
\]

And the claim follows. \hfill $\Box$

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Consider a new basis \( B = \{(1, 0), R\theta(0, 1)\} \) and an overall rotation of the system, then a general form for saddle points with eigenvalues 1 and -1 is obtained in (5), for the canonical basis in \( \mathbb{R}^2 \).

\[
A = \begin{bmatrix}
\cos(2\theta) - \sin(2\theta)\tan \psi \\
\sin(2\theta) - 2\sin^2 \theta \tan \psi \\
\sin(2\theta) + 2\cos^2 \theta \tan \psi \\
-\cos(2\theta) + \sin(2\theta)\tan \psi
\end{bmatrix}
\] (5)

It is straightforward to verify that 1 and -1 are the eigenvalues of \( A \). Now, take \(-A_2 = A_1 = A\) and observe that a convex combination of these matrices can be written as \((2\alpha - 1)A, \alpha \in [0, 1]\). Then, the characteristic polynomial is written as in (6).

\[
p(s) = s^2 - (2\alpha - 1)\text{Tr}(A)s + (2\alpha - 1)^2 \text{det}(A) = s^2 + (2\alpha - 1)^2 = (s + j(2\alpha - 1))(s - j(2\alpha - 1))
\] (6)

From (6), it can be observed that there is no strict Hurwitz combination, as the trace of \( A \) is null. However, for \( \alpha = 0.5 \), both eigenvalues are null, and the corresponding convex combination matrix is the null matrix, which is stable in the sense of Lyapunov. This is the point in Lin and Antsaklis (2009), the convex combination may be a stable matrix in the sense of Lyapunov.

Now, using the result from Lemma 1 with vector \( u \) as in (3), the condition that must be met in order to use the result from Theorem 2 (i.e., its switching law) is given in (7), for \( \varphi \in [0, 2\pi) \).

\[
\begin{bmatrix}
\cos \varphi \\
\sin \varphi
\end{bmatrix} A_i \begin{bmatrix}
\cos \varphi \\
\sin \varphi
\end{bmatrix} < 0, i \in \mathbb{N}_2 = \{1, 2\}
\] (7)

It immediately follows that, due to the fact that \( A_2 = -A_1 \), any \( \varphi \) satisfies (7), except when the inequality turns to be an equality. It can be seen that these angles are given by (8).

\[
\varphi_{eq} = \frac{1}{2} \arctg \left( \frac{\cos(2\theta) - \sin(2\theta)\tan \psi}{\sin(2\theta) + \cos(2\theta)\tan \psi} \right)
\] (8)

As the angles denoted by (8) are isolated, the result from Theorem 3 applies and the system is globally asymptotically stable if the state-dependent switching law (2) is applied.

Considering \( \theta = 30^\circ, \psi = 15^\circ \), \( x_o = [1, 1.5]' \) and simulation time of 1000s, Figure 2 is obtained, where the trajectory is seen to be going towards the origin and all arrows denoting \( \dot{x} \) are pointing to the origin, at least partially.

Therefore, this can be viewed as the extreme case to stability: there is only one convex non-stric Hurwitz combination (\( \alpha = 0.5 \)), giving the null matrix.
2.4 Example: System with Rotating Mode in $\mathbb{R}^2$

This example shows a situation that can be understood as an extension of Theorem 3. It has a rotating mode and a saddle mode. So, let:

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, $u'A_1u = 0, \forall \varphi \in [0, 2\pi)$, and:

$$u'A_2u = -\cos^2 \varphi + \sin^2 \varphi = -\cos 2\varphi < 0$$

$$\varphi \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$$

This example violates the conditions stated in Theorem 3. However, the states where $\dot{x} = 0$ are due to the center mode, which is clearly not an invariant set. Then, extending the above mentioned theorem, using La Salle’s invariance principle, it can be concluded that the system is globally asymptotically stable.

Figure 3 shows the simulation results for $x_0 = [1.5 \quad 0.5]^T$ and simulation time of 1000s.

3 Extended Results for Switched Affine Systems

3.1 The Equilibrium Point Problem

The classical definition of equilibrium point, given in books such as Khalil (1996) and Slotine and Li (1991), is a state whose time derivative is null. If the states in a given system are not redundant, matrix $A$ can be considered invertible. So, for a switched affine system written as (1), the $i$-th mode has only one equilibrium point, given by (9).

$$\dot{x}_i = -A_i^{-1}b_i$$  \hspace{1cm} (9)

This concept is extremely important, as a stable state is a subset of equilibrium points which, in most practical cases, coincide.

Then, if the desired steady-state point does not belong to the set of equilibrium points, it cannot be an stable state. However, an artificial equilibrium point can be created through switching (Bozern and Spinelli, 2004). Here, a given state will be considered a switched equilibrium point if, given a switching law, all the trajectories tend asymptotically to that state.

The desired steady-state point will be denoted by $\bar{x}$. In what follows, analogous results from the previous section will be used.

3.2 Extended Results

The same idea is present here, that is, the inner product between the state and its derivative must be negative. In this case, the law must also contain the desired steady-state point. Theorem 4 is the result obtained.

**Theorem 4** Let (1) be a switched affine system. If, for each $(x - \bar{x}) \in \mathbb{R}^n \setminus \{0\}$, $\exists i \in N_k$, such that $(x - \bar{x})^T[A_i x + b_i] < 0$, then the state-dependent switching law (10) makes the system globally asymptotically switched stable.

$$\sigma(x) = \arg \min_{i \in N_k} (x - \bar{x})^T[A_i x + b_i]$$  \hspace{1cm} (10)

**Proof:** The result immediately follows and is analogous to that in Theorem 2. The Lyapunov function to be used is $V(x) = ||x - \bar{x}||^2$.

The hypothesis of Theorem 4 is somewhat hard to verify, as there is no replacement as that in Lemma 1. Other works dealt with this problem, but in an LMI way (Mignon et al., 2000). It would be desirable to find some geometrical conditions, as it could possibly discard situations where
the Hurwitz combination of the systems is only a sufficient condition for stability.

Another point is that there could be states where the Lyapunov function derivative could be null. If these points occur isolatedly in the sense of Theorem 3, then La Salle’s invariance principle could again be used to claim global asymptotic stability.

### 3.3 Example: Buck Converter

The buck converter is probably the most important SMPS, as many other converters are derived from it. It is designed to lower DC voltages and its circuit, with the inductor’s series equivalent resistance, $r$, is shown in Figure 4.

![Buck converter](image)

Figure 4: Buck converter.

In this example, $x_1 = i_l$ is the inductor current and $x_2 = v_c$, is the capacitor voltage. Mode 1 is used for switch on and mode 2 for switch off. The matrices for each mode are:

$$A_1 = A_2 = A = \begin{bmatrix} -\frac{r}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$$

$$b_1 = b = \begin{bmatrix} V_{in} \\ 0 \end{bmatrix}, \quad b_2 = [0 \ 0]'$$

Observe that this system is composed by an affine and a linear mode, whose common $A$ matrix is such that its symmetric matrix is negative definite, as Lemma 5 proves.

**Lemma 5** In buck converter, the quadratic form of $A$ matrix is negative for all nonzero state vector if, and only if, (11) holds.

$$\frac{(L-C)^2}{LC} \leq \frac{4r}{R} \quad (11)$$

**Proof:** The result immediately follows by applying the Sylvester criterion to the symmetric part of matrix $A$, i.e., $(A + A')/2$. \qed

Moreover, the state variables are always non-negative, due to electronic parts. Now, it must be verified that conditions of Theorem 4 are met in the set:

$$\{(x_1, x_2) \in \mathbb{R}^2 / x_1, x_2 > 0\}$$

The steady-state point to be chosen is of the form in (12). The reason for including the factor $k \in \mathbb{R}^+$ and its boundaries will become clear.

$$\dot{x} = -kA^{-1}b \quad (12)$$

Now, observe that, in switch off mode, (13) holds.

$$\begin{align*}
(x - \tilde{x})' \dot{x} &= (x - \tilde{x})' A x \\
&= (x - \tilde{x})' [A(x - \tilde{x}) + A \tilde{x}] \\
&= q + (x - \tilde{x})' A \tilde{x} \\
&\overset{(12)}{=} q - k(x - \tilde{x})' b \quad (13)
\end{align*}$$

Where $q = (x - \tilde{x})' A(x - \tilde{x})$. Now, as long as $(x - \tilde{x})' b > 0$, mode off can be used. When it happens to be the opposite, activating mode on, (13) changes into (14), which can be affirmed to be negative, if $k \leq 1$.

$$\begin{align*}
(x - \tilde{x})' \dot{x} &= q + (1 - k)(x - \tilde{x})' b \\
&\quad (14)
\end{align*}$$

A simulation was made with the same parameters as in Deaecto et al. (2010): $L = 500\mu H$, $C = 470\mu F$, $R = 50\Omega$, $r = 2\Omega$ and $V_{in} = 100V$. The initial state is the origin, while the desired steady-state point is $\tilde{x} = [0.8 \ 40]'$ and 100ms were simulated. The results are in Figure 5.

![Simulation results for the buck converter](image)

Figure 5: Simulation results for the buck converter.

Observe that the sliding surface is a vertical line. This is expected, as the switching law is simplified as in (15), and $b$ is a horizontal vector.

$$\begin{align*}
(x - \tilde{x})' b \begin{cases} < 0 & \Rightarrow \text{ on mode} \\
\geq 0 & \Rightarrow \text{ off mode} \quad (15)
\end{cases}
\end{align*}$$

Another interesting point is that the result represents a great decrease in overshoot, in relation to the results obtained in Deaecto et al. (2010), but the price to pay is the time response, which greatly increases. Therefore, a trade off has
to be established between these performance parameters. Table 1 shows the comparison.

Table 1: Comparison of obtained performances.

<table>
<thead>
<tr>
<th></th>
<th>Eq. (15)</th>
<th>(Deaecto et al., 2010)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peak Current (A)</td>
<td>0.9</td>
<td>26.1</td>
</tr>
<tr>
<td>Transient time (ms)</td>
<td>89.3</td>
<td>2.5</td>
</tr>
</tbody>
</table>

4 Conclusions

In this paper, geometrical conditions for stabilizability of switched linear and affine systems were studied. For the linear ones, a simpler condition is found, which gives global asymptotic stability by just looking at any ball around the origin. Those results could be extended to systems like those proposed in Ohtake et al. (2006).

Simulations were done and the switching law for switched linear systems stabilized a bimodal two-dimensional system without any strict Hurwitz combination. In another example, a system with center mode is simulated, whose stability is guaranteed by using the well known La Salle’s invariance principle.

Geometrical conditions are much tougher to find in the case of switched affine systems, as the summing vector is constant. Moreover, in general, the system may not have an equilibrium point, so the stability concept was broadened. Finding these conditions would greatly simplify matters, especially in systems with many modes.

As an example of switched affine system, a buck converter was simulated, and its results were compared to a previous work (Deaecto et al., 2010). It was verified the great decrease in overshoot, while the response time was poor.

Finally, it can be observed the hysteresis (commonly called “chattering”) in every simulation result. This is due to the modes used: they print regions whose $\dot{x}$ vector field points to the frontier. It must be noted that they occur due to simulation step size and, in practice, to delay. However, in a theoretical study, the frontier would be more appropriately called a sliding surface.

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References


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4 The transient time is counted from rest until 98% of desired output voltage.
5 Coordenação de Aperfeiçoamento de Pessoal de Nível Superior
6 Conselho Nacional de Desenvolvimento Científico e Tecnológico